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Hitting Time and Dimension in Axiom A Systems, Generic Interval Exchanges and an Application to Birkoff Sums

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In this note we prove that for equilibrium states of axiom A systems having positive dimension the time $\tau_B(x)$ needed for a typical point x to enter for the first time in a typical ball B with radius r behaves for small r as $\tau_B(x) \sim r^{-d}$ where d is the local dimension of the invariant measure at the center of the ball. A similar relation is proved for a full measure set of interval exchanges. Some applications to Birkoff averages of unbounded (and not L^1) functions are shown.

KEY WORDS: Dimension, quantitative recurrence, Axiom A, interval exchanges, Birkoff sums.

1. INTRODUCTION

It is well known by classical recurrence results that a typical orbit of a dynamical systems comes back (in any reasonable neighborhood) near to its starting point. The quantitative study of recurrence quantifies the speed of this coming back, estimating, for example, how much time is needed to come back in any ball centered in the starting point (the reader can find and exposition of more and less recent developments about this kind of questions in the survey(7)). It turns out that in many cases the scaling law of return times is related to the dimension of the invariant measure of the system. More precisely, let us consider a starting point x, a ball $B(x, r)$ and the time $\tau_{B(x,r)}(x)$ needed for the starting point x to come back to $B(x, r)$. With these notations we have, for example,(18,19) that in exponentially mixing systems or positive entropy (with some small technical assumptions) systems over the interval $\tau_{B(x,r)}(x) \sim r^{-d(x)}$, where $d(x)$ is the local

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dimension at x.[2] Moreover[1,3] in general finite measure preserving systems the recurrence gives a lower bound to the dimension ($\lim_{r\to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log r} \le d(x)$).

A similar and related (see e.g. Ref. (16)) problem is about the time needed for a typical point y of an ergodic system to enter in some neighborhood of another point x. This leads to the *hitting time* (also called *waiting time*) indicators. For example let us denote by $\tau_{B(x,r)}(y)$ the time needed for the point y to enter in the ball $B(x, r)$ with center x and radius r (this in some sense generalizes the recurrence because we are allowed to consider an arriving point x different from the starting point y). We consider the scaling behavior of this time for small r. The hitting time indicator will have value R if $\tau_{B(x,r)}(y) \sim r^{-R}$ (precise definitions in Section 2). Again, there are relations with the local dimension. Some general relations are proved in Ref. (9) (see Theorem 2) and show that the hitting time indicator gives an upper bound to the dimension. Moreover there is a class of systems where the waiting time indicator is equal to dimension. This class of systems includes for example[9] systems having exponential distribution of return times in small balls (this includes many *more or less* hyperbolic systems over the interval, see Ref. (5)). We remark that exponential return times in balls is conjectured but (as far as we know) yet not proved in general Axiom A systems, thus equality between hitting time and dimension for axiom A does not follow from such result. A similar result, for systems on the interval preserving an absolutely continuous invariant measure was also obtained by[14].

We want to remark that there are also some relevant cases where the equality between recurrence or hitting time with dimension does not hold, hence this kind of questions are not trivial. Such cases includes rotations by Liuouville numbers (see Refs. (1, 13)), and Maps having an indifferent fixed point and infinite invariant measure[10].

A further motivation for this kind of studies is that the relations between recurrence (and similar) with dimension are used in the physical literature[12,11,15] to provide numerical methods for the study of the Hausdorff dimension of attractors. Since recurrence gives a lower bound to dimension and hitting time gives an upper bound, the combined use of these may produce efficient numerical estimators for the dimension of attractors (see also Ref. (8)).

The main result of this note is to show that in nontrivial nice examples such as Axiom A systems with positive dimension and typical Interval Exchanges

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[2] If X is a metric space and μ is a measure on X the upper local dimension at $x \in X$ is defined as $\overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \limsup_{k\in\mathbf{N}, k\to\infty} \frac{-\log(\mu(B(x,2^{-k})))}{k}$. The lower local dimension $\underline{d}_\mu(x)$ is defined in an analogous way by replacing *limsup* with *liminf*. If $\overline{d}_\mu(x) = \underline{d}_\mu(x) = d$ almost everywhere the system is called exact dimensional. In this case many notions of dimension of a measure will coincide. In particular d is equal to the dimension of the measure: $d = \inf\{\dim_H Z : \mu(Z) = 1\}$. This happen for example in systems having nonzero Lyapunov exponent almost everywhere (see for example the book in Ref. (17)).

Transformations, the hitting time indicator equals almost everywhere the local dimension $d_\mu(x)$ of the considered measure. Hence, in such systems, for typical x and y we have $\tau_{B(x,r)}(y) \sim r^{-d(x)}$. As an application of these results we give an estimation for the Birkoff averages of functions having some asymptote and no finite L^1 norm. Here the Birkoff average will increase to infinity as the number of iterations increases (this is trivially by ergodic theorem). The hitting time indicator will give an estimation about the going to infinity speed of such average (Section 5).

Since by the above result the hitting time is related to local dimension we can also (in nice systems) relate this speed to the local dimension of the invariant measure at the point where the asymptote is (see Remark 15).

2. GENERALITIES AND A CRITERIA FOR HITTING TIME AND DIMENSION

In the following we will consider a discrete time dynamical system (X, T) were X is a separable metric space equipped with a Borel finite measure μ and $T : X \to X$ is a measurable map.

Let us consider the first entrance time of the orbit of y in the ball $B(x, r)$ with center x and radius r

$$\tau_r(y, x) = \min(\{n \in \mathbf{N}, n > 0, T^n(y) \in B(x, r)\}).$$

By considering the power law behavior of $\tau_r(y, x)$ as $r \to 0$ let us define the hitting time indicators as

$$\overline{R}(y, x) = \limsup_{r\to 0} \frac{\log(\tau_r(y, x))}{-\log(r)}, \ \underline{R}(y, x) = \liminf_{r\to 0} \frac{\log(\tau_r(y, x))}{-\log(r)}[3].$$

If for some r, $\tau_r(y, x)$ is not defined then $\overline{R}(y, x)$ and $\underline{R}(y, x)$ are set to be equal to infinity. The indicators $\overline{R}(x)$ and $\underline{R}(x)$ of quantitative recurrence defined in Ref. (1) are obtained as a special case, $\overline{R}(x) = \overline{R}(x, x)$, $\underline{R}(x) = \underline{R}(x, x)$.

We recall some basic properties of $R(y, x)$:

Proposition 1. *$R(y, x)$ satisfies the following properties*

- $\overline{R}(y, x) = \overline{R}(T(y), x)$, $\underline{R}(y, x) = \underline{R}(T(y), x)$.
- *If T is $\alpha -$ Hoelder, then* $\overline{R}(y, x) \geq \alpha \overline{R}(y, T(x))$, $\underline{R}(y, x) \geq \alpha \underline{R}(y, T(x))$.
- *If we consider T^n $(n > 0)$ instead of T ;* $\overline{R}_T(y, x) \leq \overline{R}_{T^n}(y, x)$, $\underline{R}_T(y, x) \leq \underline{R}_{T^n}(y, x)$.

[3] We remark that as in the local dimension definition, since $\tau_r(y, x)$ is increasing as r decreases then $R(y, x) = \lim_{r\to 0} \frac{\log(\tau_r(y,x))}{-\log(r)} = \lim_{n\in\mathbf{N}, n\to\infty} \frac{\log(\tau_{2^{-n}}(y,x))}{n}$.

Proof: The first two points comes from Ref. (9) (and they comes directly from definitions). For the third one let us denote with τ and τ' the hitting time with respect to T and T^n. By definition $\overline{R}_T(y,x) = \text{limsup}_{r\to 0} \frac{\log(\tau_r(y,x))}{-\log(r)}$ but $\tau_r(y,x) \le n\tau_r'(y,x)$ and then $\frac{\log(\tau_r(y,x))}{-\log(r)} \le \frac{\log(\tau_r'(y,x))+\log n}{-\log(r)}$. Taking the *limsup* we are done. The same can be done for the *liminf.* □

In general systems the quantitative recurrence indicator gives only a *lower* bound on the dimension[1,3]. The waiting time indicator instead give an *upper* bound[9] to the local dimension of the measure at the point y. This is summarized in the following

Theorem 2. *If (X, T, μ) is a dynamical system over a separable metric space, with an invariant measure μ. For each x*

$$\underline{R}(y,x) \ge \underline{d}_\mu(x)\ ,\ \ \overline{R}(y,x) \ge \overline{d}_\mu(x) \tag{1}$$

holds for μ almost each y. Moreover, if X is a closed subset of $\mathbb{R}^{\mathbf{n}}$, then for almost each $x \in X$

$$\overline{R}(x,x) \le \overline{d}_\mu(x)\ ,\ \ \underline{R}(x,x) \le \underline{d}_\mu(x)\,. \tag{2}$$

As remarked in the introduction a natural question which is important also for the numerical applications is whether equality can replace the above inequalities. The following is a general criteria that ensures (together with Theorem 2) for typical points, equality between hitting time and local dimension.

Lemma 3. *Let $x \in X$ and*

$$NE_r^n(x) = X - B(x,r) \cap T^{-1}(X - B(x,r)) \cap \cdots \cap T^{-n}(X - B(x,r))$$

be the set of points that after n steps Never Enters into $B(x,r)$. If for each $\epsilon > 0$ we have $\sum \mu(NE_{2^{-n}}^{[\mu(B(x,2^{-n}))^{-1-\epsilon}]}) < \infty$ then for almost each y it holds $\overline{R}(y,x) \le \overline{d}_\mu(x)$ and $\underline{R}(y,x) \le \underline{d}_\mu(x)$.

Proof: The proof follows by a Borel Cantelli argument. Let

$$R_\epsilon = \{y \in X, \overline{R}(y,x) \ge (1+\epsilon)\overline{d}_\mu(x)\}.$$

If we prove that this set has measure zero for each ϵ we are done. If we know that $\sum \mu(NE_{2^{-n}}^{[\mu(B(x,2^{-n}))^{-1-\epsilon}]}) < \infty$ for some ϵ, this means that the set of points such that $\tau_{2^{-n}}(y,x) > [\mu(B(x,2^{-n}))^{-1-\epsilon}]$ for infinitely many n has zero measure. Taking logarithms and dividing by n we have $\frac{\log(\tau_{2^{-n}}(x,y))}{n} \le (1+\epsilon)\frac{\log(\mu(B(x,2^{-n})))}{-n}$ eventually (as n increases) for a full measure set and then $\overline{R}(y,x) = \limsup \frac{\log(\tau_{2^{-n}}(x,y))}{n} \le (1+\epsilon)\limsup \frac{\log(\mu(B(x,2^{-n})))}{-n} = (1+\epsilon)\overline{d}_\mu(x)$ on a full measure set. This is true for each ϵ and we have the statement. The same can be done for the proof of $\underline{R}(y, x) \leq \underline{d}_\mu(x)$. $\square$

3. AXIOM A SYSTEMS

In this section we will consider Axiom A systems. We will apply the properties of Gibbs measures to prove that they satisfy Lemma 3. We will prove the following

Theorem 4. *If X is a basic set of an axiom A diffeomorphism, μ is an equilibrium measure for an Hoelder potential defined on X with $\underline{d}_\mu(x) \neq 0$, then (X, T, μ) satisfies Lemma 3 at x and hence for μ almost each y it holds*

$$\underline{R}(y, x) = \underline{d}_\mu(x), \overline{R}(y, x) = \overline{d}_\mu(x).$$

We remark that such an equilibrium measure must be exact dimensional (see note 3), hence if the dimension of the measure μ is positive then $d_\mu(x) > 0$ almost everywhere.

In the proof of Theorem 4 we will use Lemma 3 approximating balls by union of cylinders of some Markov partition. The measure of cylinders is then estimated by a repeated application of the weak Bernoulli property of the equilibrium measure. This property implies that the measure of the intersection of two cylinders is near to the product of the respective measures if the time distance of such cylinders is big enough (see Eq. (3)). At each application of the Bernoulli property the estimation for the measure of the cylinder we are considering then decreases by a multiplicative factor and there is an additive error term which may be small as wanted. For this we start with a general estimation on the behavior of this kind of real sequences.

Lemma 5. *Let $0 < m < 1$ and $a_n, n \in \mathbb{N}$ be defined by*

$$\begin{cases} a_n = a_{n-1}m + \epsilon_n \\ a_0 = m^2 \end{cases}$$

where $\epsilon_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$ then for $n \geq 2$ it holds $a_n \leq \frac{m^{[\frac{n}{2}]}}{1-m} + \frac{4}{n^2}$.

Proof: We have

$$a_n = m^{n+2} + m^{n-1}\epsilon_1 + m^{n-2}\epsilon_2 + m^{n-3}\epsilon_3 + \cdots + m\epsilon_{n-1} + \epsilon_n.$$

Since $\epsilon_i < 1$ and $m < 1$ then $a_n \leq \sum_{[n/2]}^n m^i + \sum_{[n/2]}^n \epsilon_i = \frac{m^{[\frac{n}{2}]} - m^n}{1-m} + \frac{1}{([n/2])^2} - \frac{1}{([n/2]+1)^2} \leq \frac{m^{[\frac{n}{2}]}}{1-m} + \frac{4}{n^2}$. $\square$

Proof of Theorem 4: We already know that (Theorem 2) $\underline{R}(y, x) \geq \underline{d}_\mu(x)$ and $\overline{R}(y, x) \geq \overline{d}_\mu(x)$ where y varies in a full measure set. For the opposite inequalities, first we remark that (see Ref. (6) p. 72) $X = X_1 \cup \cdots \cup X_l$ where $T(X_i) = X_{i+1}$, $(T(X_l) = X_1)$ and $T^l|_{X_i}$ is topologically mixing.

By Proposition 1 we can suppose that x, y belongs to the same X_i and that T is topologically mixing (indeed replacing T with T^l we have a mixing transformation on X_i, moreover by the third point of Proposition 1 we see that if we have an upper bound for $\overline{R}_{T^l}(y, x)$ and $\underline{R}_{T^l}(y, x)$ then this is also an upper bound for $\overline{R}_T(y, x)$ and $\underline{R}_T(y, x)$. Since in this proof we are looking for an upper bound, by replacing T with T^l we can suppose that the map is topologically mixing).

To estimate the measure of the set $NE_r^n(x)$ as in Lemma 3 let us consider a Markov partition $Z = \{Z_i\}$ of X. Let $Z_m^n = T^{-m}(Z) \vee \cdots \vee T^{-n}(Z)$. By uniform hyperbolicity there are constants $C, \lambda > 0$ such that $diam(Z_{-n}^n) \leq Ce^{-\lambda n}$. By this we know that when

$$n(r) \geq -\lambda^{-1}(\log r - \log C)$$

the partition is of size so small that there is one element $\overline{Z}_0$ of the partition $\overline{Z} = Z_{-n(r)}^{n(r)}$ which is included in $B(x, r)$.

Now $NE_r^n(x) \subseteq B_0^n = X - \overline{Z}_0 \cap T^{-1}(X - \overline{Z}_0) \cap \cdots \cap T^{-n}(X - \overline{Z}_0)$. We remark that B_0^n is the union of many cylinders and the measure of B_0^n decreases very fast because of the properties of the equilibrium measure μ. Indeed by Ref. (6) (see p. 90) we know that since the map T can be supposed to be topologically mixing then μ has the weak Bernoulli property: i.e. let us consider $t, s \geq 0$ and the partitions $P_s = Z \vee T^{-1}(Z) \vee \cdots \vee T^{-s}(Z)$ and $Q_t = T^{-t}(Z) \vee \cdots \vee T^{-t-k}(Z)$. For each ϵ, if the time distance $t - s = N_Z(\epsilon)$ is big enough, then the partitions behaves as independent up to any prescribed error ϵ:

$$\sum_{P \in P_s, Q \in Q_t} \mu(Q \cap P) - \mu(P)\mu(Q) < \epsilon. \tag{3}$$

Moreover by Ref. (6), Theorem 1.25 we can find an estimation for $N_Z(\epsilon)$ as a function of ϵ (see Ref. (6), p. 38): $N_Z(\epsilon) = -c\log(\epsilon) + c'$, where c, c' are constants depending on μ, T and Z.

To estimate $\mu(B_0^l)$ by cylinders of $\overline{Z}$ we now remark the fact that a non empty cylinder for the partition $\overline{Z} = Z_{-n(r)}^{n(r)}$ is also a cylinder for the partition Z. Indeed the cylinder $\overline{\mathbf{z}}_m = \overline{Z}_{i_1} \cap T^{-1}(\overline{Z}_{i_2}) \cap \cdots \cap T^{-m-1}(\overline{Z}_{i_m})$, $\overline{Z_i} \in \overline{Z}$ satisfies $\overline{\mathbf{z}}_m = \mathbf{z}_{m+2n}$ where $\mathbf{z}_{m+2n} = T^n(Z_{j_1}) \cap T^{n-1}(Z_{j_2}) \cap \cdots \cap T^{-m-1-n}(Z_{j_{(m+2n)}})$ is a cylinder of Z and $\overline{Z}_{i_k} = T^{-k+n}Z_{j_k} \cap \cdots \cap T^{-k-n}Z_{j_{k+2n}}$. Since μ is preserved then $\mu(T^{-k+n}Z_{j_k} \cap \cdots \cap T^{-k-n}Z_{j_{k+2n}}) = \mu(T^{-k}Z_{j_k} \cap \cdots \cap T^{-k-2n}Z_{j_{k+2n}})$ hence we can apply the weak Bernoulli property to such cylinders obtaining that also $\overline{Z}$ satisfies such a property and

$$N_{\overline{Z}}(\epsilon) \leq -c\log(\epsilon) + c' + 2n \tag{4}$$

(where c, c' are the constants of $N_Z(\epsilon)$ as above). We recall that n depends on r and we can choose $n(r) \leq -\lambda^{-1}(\log r - \log C)$.

By Eq. (4), the measure of a long cylinder will be estimated by applying many times the Bernoulli property considering a sequence of errors $\epsilon(n)$ as in Lemma 5 and the corresponding $N_Z(\epsilon(n))$.We will estimate the measure of a long cylinder $\overline{Z}_{i_1} \cap T^{-1}(\overline{Z}_{i_2}) \cap \cdots \cap T^{-m-1}(\overline{Z}_{i_m})$ by considering it included in the intersection of subcylinders given by

$$\overline{Z}_{i_1} \cap T^{-N_Z(\epsilon(1))}\left(\overline{Z}_{i_{N_Z(\epsilon(1))}}\right) \cap T^{-N_Z(\epsilon(1))-N_Z(\epsilon(2))}\left(\overline{Z}_{i_{N_Z(\epsilon(1))+N_Z(\epsilon(2))}}\right), \ldots$$

whose time distance from the previous to the next intersecting cylinder is such that Eq. (3) can be applied with the given errors $\epsilon(n)$. Let us then apply the weak Bernoulli property of $\overline{Z}$ to get an estimation for $\mu(B_0^l)$. Let us set $m = \mu(X - Z_0)$ and $\epsilon(i) = \frac{2i+1}{i^2(i+1)^2} = \frac{1}{i^2} - \frac{1}{(i+1)^2}$. We have (Eq. 4) that setting $C'(r) = c' - 2\lambda^{-1}(\log r - \log C)$ then

$$N_{\overline{Z}}(\epsilon(i)) \leq -c\log(\epsilon(i)) + c' + 2n(r) = -c\log\left(\frac{2i+1}{i^2(i+1)^2}\right) + C'(r)$$

and for $i > 1$ there is a C_2 s.t. $N_{\overline{Z}}(\epsilon(i)) \leq C_2\log(i) + C'(r)$.

Let us set $\eta_i = \sum_{j\leq i} N_{\overline{Z}}(\epsilon(j))$. To compact formulas we remark that when r is small, for each δ there is a K not depending on r such that, if i is big enough

$$\eta_i \leq -Ki^{1+\delta}\log r. \tag{5}$$

As said before, the measure of $B_0^{\eta_i}$ can then be estimated applying i times the weak Bernoulli property, with $\epsilon(i) = \frac{2i+1}{i^2(i+1)^2}$ as above, to subcylinders of increasing length $N_{\overline{Z}}(\epsilon(1))$, $N_{\overline{Z}}(\epsilon(1)) + N_{\overline{Z}}(\epsilon(2))$, $N_{\overline{Z}}(\epsilon(1)) + N_{\overline{Z}}(\epsilon(2)) + N_{\overline{Z}}(\epsilon(3)) \cdots$ obtaining by the Bernoulli property of μ

$$\mu\left(B_0^{N(\epsilon(1))}\right) \leq m^2 + \epsilon(1),$$

$$\mu\left(B_0^{N(\epsilon(1))+N(\epsilon(2))}\right) \leq (m^2 + \epsilon(1))m + \epsilon(2),$$

$$\mu\left(B_0^{N(\epsilon(1))+N(\epsilon(2))+N(\epsilon(3))}\right) \leq ((m^2 + \epsilon(1))m + \epsilon(2))m + \epsilon(3), \ldots$$

Hence by Lemma 5 above

$$\mu\left(B_0^{\eta_i}\right) \leq \frac{m^{[\frac{i}{2}]}}{1-m} + \frac{4}{i^2}.$$

We remarked that $NE_r^n \subset B_0^n$. If we consider another element Z_1 of $\overline{Z}$ with $Z_1 \subset B(x,r)$ and $B_1^n = X - (Z_0 \cup Z_1) \cap T^{-1}(X - (Z_0 \cup Z_1)) \cap \cdots \cap T^{-n}(X - Z_0 \cup Z_1))$, we have also $NE_r^n \subset B_1^n \subset B_0^n$. Now considering a sequence $Z_0, \ldots, Z_w$ of elements of $\overline{Z}$ with $Z_0, \ldots, Z_w \subset B(x,r)$ and $B_w^n = X \cap T^{-1}(X - (Z_0 \cup \cdots \cup Z_w)) \cap \cdots \cap T^{-n}(X - (Z_0 \cup \cdots \cup Z_w))$, we have also $NE_r^n \subset B_w^n$.

The measure of B^n_w can be estimated as above, obtaining $\mu(B^{\eta_i}_w) \le \frac{m_w^{[i/2]}}{1-m_w} + \frac{4}{i^2}$, where $m_w = \mu(X - (Z_0 \cup \cdots \cup Z_w))$.

Now, refining again the partition $\overline{Z}$ if necessary (as before this is possible because the diameter of each element of Z_n^{-n} is less or equal than $Ce^{-\lambda n}$, and this will only change the constants in $N_{\overline{Z}}(\epsilon)$) we can suppose that each piece of the partition has diameter less than $r/4$. We then have that we can choose $Z_0, \ldots, Z_w$ such that $B(x, \frac{r}{2}) \subset Z_0 \cup \cdots \cup Z_w \subset B(x, r)$. Then $\mu(X - (Z_0 \cup \cdots \cup Z_w)) \le \mu(X - B(x, \frac{r}{2}))$. This gives,

$$\mu\left(B^{\eta_i}_w\right) \le \frac{\left(1 - \mu\left(B\left(x, \frac{r}{2}\right)\right)\right)^{[\frac{i}{2}]}}{\mu\left(B\left(x, \frac{r}{2}\right)\right)} + \frac{4}{i^2}.$$

By Eq. (5) we then have $i \ge \frac{\eta_i^{1/1+\delta}}{(K \log r/4)^{1/1+\delta}}$, by this, setting $r = 2^{-n}$

$$\begin{aligned}\mu\left(NE_{2^{-n}}^{[\mu(B(x,2^{-n}))^{-1-\epsilon}]}\right) &\le \mu\left(B_w^{[\mu(B(x,2^{-n}))^{-1-\epsilon}]}\right)\\ &\le \frac{(1 - \mu(B(x, 2^{-n-1})))^{\left[\frac{1}{2}(Kn+\log 4)^{-1/1+\delta}\mu(B(x,2^{-n}))^{-1-\epsilon/1+\delta}\right]}}{\mu(B(x, 2^{-n-1}))}\\ &\quad + \frac{4}{(Kn + \log 4)^{-2/1+\delta}\mu(B(x, 2^{-n}))^{-2-2\epsilon/1+\delta}}.\end{aligned}$$

When n is big, recalling that δ can be chosen as small as we want and hence smaller than ϵ, then $\mu(NE_{2^{-n}}^{\mu(B(x,2^{-n}))^{-1-\epsilon}})$ is less than about $\frac{(e^{-1})^{Kn^{\frac{-1}{1+\delta}}\mu(B(x,2^{-n}))^{-\epsilon+\delta/1+\delta}}}{\mu(B(x,2^{-n-1}))} + 4(Kn + \log 4)^{2/1+\delta}\mu(B(x, 2^{-n-1}))^{2+2\epsilon/1+\delta}$ since $\underline{d}(x) > 0$ then $\mu(B(x, 2^{-n}))$ decreases exponentially fast and we have $\sum_n \mu(NE_{2^{-n}}^{\mu(B(x,2^{-n}))^{-1-\epsilon}}) < \infty$. This is enough to apply the Lemma 3 and have the required statement. $\square$

4. INTERVAL EXCHANGES

Interval Exchanges are particular bijective piecewise isometries which preserves the Lesbegue measure. In this section we apply a result of Boshernitzan about a full measure class of uniquely ergodic interval exchanges to prove equality between hitting time and dimension for almost each point. We refer to Ref. (4) for generalities on this important class of maps.

Let T be some interval exchange. Let $\delta(n)$ be the minimum distance between the discontinuity points of T^n. We say that T has the property $\tilde{P}$ if there is a constant C and a sequence n_k such that $\delta(n_k) \ge \frac{C}{n_k}$. Let us recall the above cited result and some facts we are going to use in this section:

Lemma 6. *(by Ref. (4)) The set of interval exchanges having the property $\tilde{P}$ has full measure in the space of interval exchange maps.*

Lemma 7. *If T is an interval exchange then for almost each point x then*

$$\underline{R}(y,x) = \underline{R}(y,T(x)) \text{ and } \overline{R}(y,x) = \overline{R}(y,T(x))$$

holds for almost each y.

Proof: If x and $T(x)$ are not discontinuity points then T is an isometry between a small neighborhood of x and a small neighborhood of $T(x)$. Thus, when $d(T^j(y), x)$ is small it holds $d(T^j(y), x) = d(T^{j+1}(y), T(x))$ and the statement then follow directly from the definition of $\overline{R}(y,x)$ and $\underline{R}(y,x)$. □

Lemma 8. *Let T be an interval exchange having the property $\tilde{P}$. There is a positive measure set $B \subset [0,1]$ such that for each $x \in B$ there is a subsequence n_{k_i} such that for each i*

$$\min_{h,j\le \frac{n_{k_i}}{2},\ h\neq j} d(T^{-h}(x), T^{-j}(x)) \ge \frac{C}{2n_{k_i}}, \text{ and } \min_{h\le \frac{n_{k_i}}{2}} d(T^{-h}(x), x_0) \ge \frac{C}{4n_{k_i}}$$

for each discontinuity point x_0.

Proof: Let x_0 be a discontinuity point. Let us consider the sets

$$J_k = \cup_{i\le \frac{n_k}{2}} \left[T^{-i}(x_0) + \frac{C}{4n_k}, T^{-i}(x_0) + \frac{C}{2n_k}\right]$$

if T has the property $\tilde{P}$ these intervals are disjoint and then $\mu(J_k) \ge \frac{C}{8}$. Moreover, for each $i \le n_k$ the interval $[T^{-i+1}(x_0), T^{-i+1}(x_0) + \frac{C}{2n_k}]$ is mapped by T^{-1} isometrically onto $[T^{-i}(x_0), T^{-i}(x_0) + \frac{C}{2n_k}]$. This is because still by property $\tilde{P}$ for $i \le n_k$ all T^{-i} iterates of x_0 are at a distance greater than $\frac{C}{n_k}$ from all the discontinuity points (included x_0 itself). Hence if $x \in J_k$ then $\min_{h,j\le\frac{n_k}{2},h\neq j} d(T^{-h}(x), T^{-j}(x)) \ge \frac{C}{2n_k}$ and $d(x,x_0) \ge \frac{C}{4n_k}$, moreover if x_i is a discontinuity point for T then $\min_{h\le\frac{n_k}{2}} d(T^{-h}(x), x_i) \ge \delta(n_k) - \frac{C}{2n_k} \ge \frac{C}{2n_k}$. Since $\mu(J_k) \ge \frac{C}{8}$ there is a positive measure set B of points such that $x \in B$ implies that x belongs to infinitely many J_k and hence for those x there is a subsequence n_{k_i} as in the statement. □

Theorem 9. *If an interval exchange transformation T is ergodic and has the property $\tilde{P}$ then for almost each point x it holds $\underline{R}(y,x) = 1$ for almost each $y \in [0,1]$.*

Proof: The inequality $\underline{R}(y,x) \geq 1$ follows by Theorem 2. For the other inequality, let $x \in B$ (as in Lemma 8), let us consider the following $L_j = \cup_{i \leq \frac{1}{2} n_{k_j}} B(T^{-i}(x), \frac{C}{8 n_{k_j}})$ similar to before, by Lemma 8 this is made of disjoint intervals and $\mu(L_j) \geq \frac{C}{16}$ hence there is a positive measure set B' of points belonging to infinitely many L_j. Moreover if $y \in L_j$ then $y \in B(T^{-i}(x), \frac{C}{8 n_{k_j}})$ with $i \leq \frac{1}{2} n_{k_j}$. By Lemma 8 since $\min_{h \leq \frac{n_{k_i}}{2}} d(T^{-h}(x), x_0) \geq \frac{C}{4 n_{k_i}}$ for each discontinuity point x_0 we have that $d(T^i(y), x) \leq \frac{C}{8 n_{k_j}}$. If $y \in B'$ then there is a sub sequence $n_{k_{j_i}}$ giving $\tau_{\frac{C}{8 n_{k_{j_i}}}}(y,x) \leq \frac{1}{2} n_{k_{j_i}}$. This implies that $\underline{R}(y,x) \leq 1$ for $x \in B$ and y varying in a positive measure set B' and hence by ergodicity for almost each y. By Lemma 7 and ergodicity we conclude that the statement hold for almost each x. $\square$

Remark. With the same proof as above, using property $\tilde{P}$ directly on discontinuity points we can also obtain.

Proposition 10. *If T has the property $\tilde{P}$, for each discontinuity point x_0 it holds $\underline{R}(y,x_0) = 1$ for almost each $y \in [0,1]$.*

In interval exchanges the only source of initial condition sensitivity is the discontinuity (the orbits of two points can be only separated by a discontinuity) we remark that an estimation of the approaching speed of typical orbits to the discontinuity is useful to estimate the kind of initial condition sensitivity and the kind of "weak" chaos that is present is such maps. The theorem above in some sense can give (using the construction done in Ref. (2)) an upper bound on the initial condition sensitivity of interval exchanges. We will not go into details about this point in this work, however.

5. HITTING TIME AND BIRKOFF SUMS

Let (X, T) a measure preserving transformation on a metric space X, $x_0 \in X$ and let us consider a measurable function $f : X - \{x_0\} \to \mathbb{R}$ satisfying the following

- f is bounded outside each neighborhood of x_0 and $f \geq 0$
- $0 < \lim_{x \to x_0} \frac{f(x)}{d(x,x_0)^{-\alpha}} < \infty$ (f has a vertical asymptote in x_0 where $f(x) \sim d(x,x_0)^{-\alpha}$)
- $\alpha > \overline{d_\mu}(x_0)$ (which implies $\int_X f d\mu = \infty$).

By the ergodic theorem we know that for almost each x the Birkoff average $\frac{S_n(x)}{n} = \frac{1}{n} \sum_{i=0}^{n} f(T^i(x))$ is such that $\frac{S_n(x)}{n} \to \infty$. We can have an estimation for the speed of increasing of $\frac{S_n(x)}{n}$ by the behavior of the hitting time indicator $R(x, x_0)$.

Theorem 11. *Let us suppose X, x_0, f as above then for each x*

$$\frac{\alpha}{\underline{R}(x, x_0)} \leq \limsup_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{\underline{R}(x, x_0)} + 1$$
$$\frac{\alpha}{\overline{R}(x, x_0)} \leq \liminf_{n\to\infty} \frac{\log(S_n(x))}{\log(n)}.$$

Before the proof, let us recall a result from Ref. (2) which is also related to Theorem 2.

Lemma 12. *Let (X, μ, d, T) be a measure preserving transformation on a metric space, and, $x \in X$. If $\beta > \frac{1}{\underline{d}_\mu(x)}$ then for μ-almost every $y \in X$ we have*

$$\liminf_{n\to\infty} n^\beta \cdot d(x, T^n y) = \infty.$$

Proof of Theorem 11: By Lemma 12 it holds that for each $\epsilon > 0$, if n is big enough $d(T^n(y), x_0) \geq n^{-\frac{1}{\underline{R}(y,x_0)}-\epsilon}$. Now we remark that by the assumptions on f there are c_1, c_2 such that $f(x) \leq \max(c_1, c_2 d(x, x_0)^{-\alpha})$. Then if n is big enough

$$\sum_{i=0}^{n} f(T^i(y)) \leq \sum_{i=0}^{n} \max(c_1, c_2 d(T^i(y) - x_0)^{-\alpha})$$
$$\leq \sum_{i=0}^{n} \max(c_1, c_2 n^{\alpha/\underline{R}(y,x_0)+\alpha\epsilon})$$

and we have $\limsup_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \leq \frac{\alpha}{\underline{R}(y,x_0)} + 1$. To prove the lower bounds we remark that by definition of $\overline{R}(y, x_0)$ we obtain that eventually $\tau_r(y, x_0) \leq r^{-\overline{R}(y,x_0)-\epsilon}$ and then for each small r there is $n_r \leq r^{-\overline{R}(y,x_0)-\epsilon}$ such that $d(T^{n_r}(y), x_0) \leq r$. Let us consider a sequence $r_m = m^{1/-\overline{R}(y,x_0)-\epsilon}$, then $n_{r_m} \leq m$ and we have that $\min_{i\leq m}(T^i(y), x_0) \leq r_m$ and then eventually $\sum_{i=0}^{n} f(T^i(y)) \geq c_3 n^{\alpha/\overline{R}(y,x_0)+\epsilon}$, which gives $\frac{\alpha}{\overline{R}(x,x_0)} \leq \liminf_{n\to\infty} \frac{\log(S_n(x))}{\log(n)}$. On the other hand, by the definition of $\underline{R}(y, x_0)$ we have that $\forall\epsilon$ there is a sequence of times n_i s.t. $d(T^{n_i}(y), x_0) \leq n_i^{-1/\underline{R}(y,x_0)+\epsilon}$, then there is some c_3 such that frequently $\sum_{i=0}^{n} f(T^i(y)) \geq c_3 n^{\alpha/\underline{R}(y,x_0)-\alpha\epsilon}$, which proves the last statement left. □

We end with some upper bound on the behavior of $\liminf_{k\to\infty}\frac{\log(S_n)}{\log n}$. Now we give a general Lemma which is used in the following, because of its generality it is also interesting by itself.

Lemma 13. *Let $\epsilon, c > 0$ and $a_k = \frac{c}{k^{1/\underline{d}_\mu(x)+\epsilon}}$. Let us consider the truncated functions*

$$f_k(x) = \begin{cases} f(x) \text{ if } d(x, x_0) > a_k \\ 0 \text{ if } d(x, x_0) \le a_k \end{cases},$$

with the L^1 norm $\|f_k\| = \int_X f_k d\mu$. Then, if c is small enough, there is a set A such that $\mu(A) \ge \frac{1}{4}$ and for all $x \in A$

$$\liminf_{k\to\infty} \frac{S_k(x)}{\|f_1\| + \|f_2\| + \cdots + \|f_{k-1}\| + \|f_k\|} \le 4.$$

Proof: Let $A_c = \{x \in X, s.t. \forall n \ge 0 \ T^n(x) \notin B(x_0, a_n)\}$. By the above Lemma 12 if c is smaller and smaller then $\mu(A_c) \to 1$, hence there is a $c > 0$ such that $\mu(A_c) \ge \frac{3}{4}$.

Let us consider $\tilde{f}_k(x) = \frac{f_1(x)+f_2(T(x))+\cdots+f_k(T^{k-1}(x))}{\|f_1\|+\|f_2\|+\cdots+\|f_k\|}$, since T is measure preserving this is a sequence of functions whose L^1 norm is 1.

Let us consider $\underline{\tilde{f}}(x) = \liminf_{k\to\infty}(\tilde{f}_k(x))$. By the Fatou Lemma $\|\underline{\tilde{f}}(x)\| \le 1$ and then $\underline{\tilde{f}}(x) \le 4$ on a set B whose measure is such that $\mu(B) \ge \frac{3}{4}$. We remark that if $x \in A_c$ then for each k

$$f_1(x) + f_2(T(x)) + \cdots + f_k(T^{k-1}(x)) = f(x) + f(T(x)) + \cdots + f(T^{k-1}(x)).$$

Then $\underline{\tilde{f}}_k = \liminf_{k\to\infty} \frac{f(x)+\cdots+f(T^{k-1}(x))}{\|f_1\|+\cdots+\|f_k\|}$ on A_c and then $\liminf_{k\to\infty} \times \frac{f(x)+\cdots+f(T^{k-1}(x))}{\|f_1\|+\cdots+\|f_k\|} \le 4$ on the set $A = A_c \cap B$ whose measure is greater than $\frac{1}{4}$. □

We remark that in the above proof T is not supposed to be ergodic.

Theorem 14. *If X is an n dimensional manifold and the invariant ergodic measure μ is absolutely continuous, with a bounded density in a neighborhood of x_0 and $d_\mu(x_0) = n$, then for a.e. x*

$$\liminf_{k\to\infty} \frac{\log(S_k(x))}{\log k} \le \frac{\alpha}{n}.$$

Proof: Let f_k be as in Lemma 13. Let us estimate the L^1 norms $\|f_k\|$. For this let us consider the measure of the set $J_{\Delta r} = B(x_0, r + \Delta r) - B(x_0, r)$, by the assumptions on μ there is a K_1 such that when Δr is small we have that the measure of $J_{\Delta r}$ is less or equal than $K_1 r^{n-1}\Delta r$.

Since there is some K_2 such that $f(x) \le K_2 d(x, x_0)^{-\alpha}$, setting $r = d(x, x_0)$ we have, recalling that $\alpha > n$

$$\int_X f_k d\mu \le \int_{a_k}^{\infty} K_1 K_2 r^{-\alpha} r^{n-1} dr = K_1 K_2 \int_{a_k}^{\infty} r^{n-\alpha-1}\, dr = \frac{-K_1 K_2}{n-\alpha} a_k^{n-\alpha}$$

By the definition of a_k then there are constants C_1, C_2 such that

$$\|f_k\| \le \frac{-K_1 K_2}{n-\alpha} \left(\frac{c}{k^{\frac{1}{n}+\epsilon}}\right)^{n-\alpha} \le C_1 k^{-1-\epsilon n + \frac{\alpha}{n} + \epsilon\alpha}$$

and

$$\sum_{i=1}^{k} \|f_k\| \le \sum_{i=1}^{k} i^{-1-\epsilon n + \frac{\alpha}{n} + \epsilon\alpha} \le C_2 k^{-\epsilon n + \frac{\alpha}{n} + \epsilon\alpha}$$

then, applying the Lemma 13 remembering that we can take ϵ to be as small as wanted we have for $x \in A$

$$\liminf_{k\to\infty} \frac{f(x) + \cdots + f(T^{k-1}(x))}{k^{-\epsilon n + \frac{\alpha}{n} + \epsilon\alpha}} \le 4C_1 C_2$$

and then $\liminf_{k\to\infty} \frac{\log(f(x)+\cdots+f(T^{k-1}(x)))}{\log k} \le -\epsilon n + \frac{\alpha}{n} + \epsilon\alpha$ for each ϵ on the set A having positive measure. The result follows by the ergodicity of μ. $\square$

Remark 15. By the above results and the ones in the previous sections it easily follows:

1. (by Theorem 2) In a general system, if the local dimension at x_0 is $d_\mu(x_0)$. Then for almost each x

$$\limsup_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \le \frac{\alpha}{d_\mu(x_0)} + 1$$

2. (by Theorems 9 and 14) If T is an IET and x_0 is typical or a discontinuity point then for almost each x

$$\alpha = \liminf_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \le \limsup_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \le \alpha + 1$$

3. (by Theorem 4) If (X, T) is axiom A (with an equilibrium measure, as in Theorem 4), x_0, x are typical and $d \ne 0$ is the dimension of the measure then

$$\frac{\alpha}{d} \le \liminf_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \le \limsup_{n\to\infty} \frac{\log(S_n(x))}{\log(n)} \le \frac{\alpha}{d} + 1.$$

Remark 16. We remark that using Eq. (2) we can obtain as above similar lower bounds for the behavior of $S_n(x_0)$ when x_0 is a typical point.

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